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AN ANALYSIS OF THE TRANSIENT EDGE EFFECT ON HEAT CONDUCTION IN WEDGES

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NOMENCLATURE

r,	radial coordinate;
s,	Laplace transform operator;
t,	time;
T,	temperature;

Greek symbols

α,	thermal diffusivity;
$\Delta(\theta_0)$,	wedge invariant;
θ,	angular coordinate;
θ_0,	wedge angle;
λ,	$= (s/\alpha)^{1/2}$;
ϕ,	Laplace-transformed temperature;
ψ,	linearly transformed temperature.

Functions

$f(r)$,	deviation function of $\psi(r,\theta)$;
$g(r)$,	deviation function of $\phi(r,\theta)$.

INTRODUCTION

THE CLASSICAL problem of the heat conduction in an infinite wedge is considered in a generalized sense. The basis for the treatment involved hinges upon the assumptions that a uniform initial temperature for the wedge enclosure and a constant surface temperature are prescribed *a priori*, and that the thermophysical properties of the homogeneous enclosure are independent of temperature. The first analytical solution for the time-dependent temperature field in such a wedge was due to Jaeger [1]. Jaeger's solution involves only two parameters, the angular coordinate and a Fourier number, but consists of an infinite series of special functions. More recently, Budhia and Kreith [2] discussed the same problem as a special case in their treatment of the heat transfer with melting or freezing in a wedge, using the Green's function given by Carslaw and Jaeger [3].

Two interesting observations have been made on rectangular wedges. Carslaw and Jaeger [3] solved the case of an external rectangular corner (a wedge angle of 90°) via a product solution approach and showed that the difference between the rate of loss of heat per unit time, per unit depth along the two bounding planes, from the corner surface and that without the edge effect, i.e. from a corresponding surface of a semi-infinite solid, is independent of the time variable.* The same invariability of the rate of loss of heat over an internal rectangular corner (a wedge angle of 270°) has been observed experimentally by Ruddle and Skinner [4] in their study of the heat extraction at corners in sand molds filled with molten aluminum–30% copper alloy, although they did not point out the invariability in explicit terms.†

This paper is directed towards the generalization of the invariability for non-rectangular wedges. The analytical solutions mentioned above [1, 2] do not lend themselves to a direct investigation of the generalized invariability. The approach presented in this paper transforms the time-dependent problem into a steady diffusion problem for which a closed form solution is available [5]. By inversely transforming the steady diffusion solution into the space and time domain, it is shown that the invariability of the rate of loss of heat by conduction applies to all wedges.

ANALYSIS

The following analysis is intended to indicate the general nature of the invariability of transient heat conduction in wedges. It is to be shown that the integrated edge effect, defined as the difference between the rate of loss of heat per unit time, per unit depth along the two bounding planes, from the wedge surface and that from a corresponding surface of a semi-infinite solid, is independent of time for all wedges.

An infinite wedge enclosure bounded by two half-planes is the domain of present interest. The space is represented by a cylindrical coordinate system as shown in Fig. 1 with the z-axis being perpendicular to the page. The analytical solution of the 2-dim. heat conduction equation (1), complying with the *a priori* conditions (2), can be used to evaluate the local heat transfer rate per unit area at the wedge surfaces $\theta = 0$ or $\theta = \theta_0$.

$$\nabla^2 T(r,\theta,t) = \frac{1}{\alpha}\frac{\partial}{\partial t} T(r,\theta,t) \tag{1}$$

with

$$T(r,\theta,0) = 0, \quad T(r,0,t) = 1, \quad T(r,\theta_0,t) = 1. \tag{2}$$

As was mentioned previously, the currently available closed form solutions [1, 2] for the transient temperature field $T(r,\theta,t)$ in the above problem, do not lend themselves to a direct investigation of the generalized invariability.

An alternative approach is to first Laplace-transform equations (1) and (2) with respect to the time t and obtain

$$\nabla^2 \phi(r,\theta) - \frac{s}{\alpha}\phi(r,\theta) = 0 \tag{3}$$

with

$$\phi(r,0) = \frac{1}{s}, \quad \phi(r,\theta_0) = \frac{1}{s} \tag{4}$$

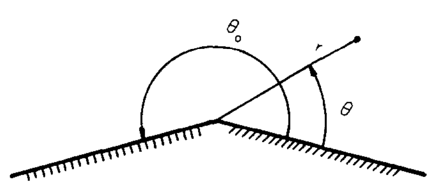

FIG. 1. Coordinates utilized in problem formulation ($0 < \theta_0 < 2\pi$).

*See p. 172 of [3].
†See Fig. 5 of [4].

where s is the Laplace transform operator and the Laplace-transformed temperature $\phi(r,\theta)$ contains s as a parameter.

The above equations can be further simplified by a rearrangement of variables. Let

$$\psi(r,\theta) = \alpha\left[\phi(r,\theta) - \frac{1}{s}\right] \quad (5)$$

then

$$\nabla^2\psi(r,\theta) - \lambda^2\psi(r,\theta) = 1 \quad (6)$$

with

$$\psi(r,0) = 0, \quad \psi(r,\theta_0) = 0 \quad (7)$$

where $\lambda = (s/\alpha)^{1/2}$.

Equation (6) with given *a priori* conditions (7) appeared previously in a steady diffusion problem associated with neutron absorption in control rods embedded within a medium where neutron diffusion length is $1/\lambda$ and was solved independently by Hurwitz and Roe [6], and Levine [5]. Levine's approach, based on a pair of integral transform relations, the so-called Grünberg modification of the Kontorovich–Lebedev reciprocal formulas [5], is followed and the solution for $\psi(r,\theta)$ in the domain $0 < \theta < \theta_0$, $0 < r < \infty$, subject to the conditions (7), is

$$\psi(r,\theta) = -\frac{1}{\lambda^2} + \frac{1}{2\lambda^2\theta_0}\int_{-\infty}^{\infty} e^{-i\lambda r\sinh\eta} \times \left(\operatorname{cosec}\pi\frac{\theta+i\eta}{\theta_0} + \operatorname{cosec}\pi\frac{\theta-i\eta}{\theta_0}\right)d\eta. \quad (8)$$

It follows that $T(r,\theta,t)$ can be obtained by an inverse Laplace transform of $\phi(r,\theta)$, obtainable from equation (8), and that the normal derivatives of $\psi(r,\theta)$ at $\theta = 0$ and $\theta = \theta_0$ are symmetric with respect to the origin. Expression (8) can be used to calculate the normal derivative of $\psi(r,\theta)$ at $\theta = 0$.

$$\left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{\theta=0} = -\frac{1}{\lambda\theta_0}\int_{-\infty}^{\infty}\frac{\cosh\eta}{\sinh\frac{\pi\eta}{\theta_0}}\sin(\lambda r\sinh\eta)\,d\eta. \quad (9)^*$$

Equation (9) yields a special solution for $\theta_0 = \pi$, which corresponds to a simple 1-dim. problem:

$$\left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{\theta=0,\ \theta_0=\pi} = -\frac{1}{\lambda}. \quad (10)$$

Levine then defined a useful difference function

$$f(r)^\dagger = \left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{\theta=0} - \left.\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right|_{\theta=0,\ \theta_0=\pi} \quad (11)$$

and obtained a deviation integral

$$\int_0^\infty f(r)\,dr = \frac{\alpha}{s}\int_{-\infty}^{\infty}\coth\eta\left(\frac{1}{\pi\sinh\eta} - \frac{1}{\theta_0\sinh\frac{\pi\eta}{\theta_0}}\right)d\eta \quad (12)^\ddagger$$

with $1/\lambda^2$ being replaced by α/s.

The above result enables one to define a similar difference function of the normal derivative of $\phi(r,\theta)$ at $\theta = 0$ as

$$g(r)^* = \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{\theta=0} - \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{\theta=0,\ \theta_0=\pi} \quad (13)$$

and obtain

$$\int_0^\infty g(r)\,dr = \frac{1}{s}\int_{-\infty}^{\infty}\coth\eta\left(\frac{1}{\pi\sinh\eta} - \frac{1}{\theta_0\sinh\frac{\pi\eta}{\theta_0}}\right)d\eta. \quad (14)$$

A physical interpretation for expression (14) is essential. It represents, over a semi-infinite strip with unit depth in the z-direction across half the wedge surface, the integrated difference of the normal derivative of the Laplace-transformed temperature at the wedge surface with respect to that without the edge effect ($\theta_0 = \pi$). The separation of the Laplace transform operator s from the integral in (14) renders a simple inverse Laplace transform into the original space and time domain. One can define a corresponding deviation integral as

$$\int_0^\infty\left(\left.\frac{1}{r}\frac{\partial T}{\partial\theta}\right|_{\theta=0} - \left.\frac{1}{r}\frac{\partial T}{\partial\theta}\right|_{\theta=0,\ \theta_0=\pi}\right)dr = \int_{-\infty}^{\infty}\coth\eta\left(\frac{1}{\pi\sinh\eta} - \frac{1}{\theta_0\sinh\frac{\pi\eta}{\theta_0}}\right)d\eta \quad (15)$$

which represents the integrated edge effect over the surface $\theta = 0$.

The fact that the right-hand side of (15) is independent of time t is striking. This leads to the definition of a deviation integral, over the two wedge surfaces per unit depth in the z-direction, that is a function of θ_0 only. The time-independent deviation integral is denoted by $\Delta(\theta_0)$ and

$$\Delta(\theta_0) = 2\int_{-\infty}^{\infty}\coth\eta\left(\frac{1}{\pi\sinh\eta} - \frac{1}{\theta_0\sinh\frac{\pi\eta}{\theta_0}}\right)d\eta. \quad (16)$$

The above deviation integral was termed, by Levine in solving the steady diffusion problem for an effective neutron absorption area [5], the corner correction function. However, in order to emphasize the extended utility as depicted by expression (15), it is felt that here $\Delta(\theta_0)$ should be called the wedge invariant. A graphical representation for $\Delta(\theta_0)$ vs θ_0 was given by Hurwitz and Roe [6] and is reproduced in Fig. 2. The theoretical basis for the invariability of the rate of loss of heat by conduction, for all wedges, is thus established.

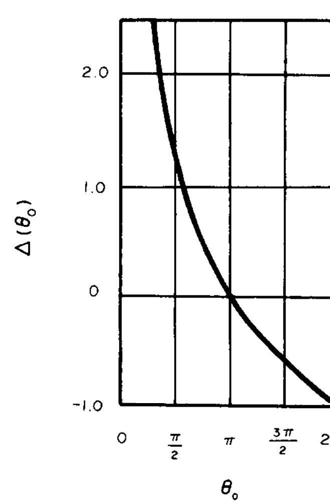

FIG. 2. Plot of $\Delta(\theta_0)$ vs θ_0; after Hurwitz and Roe [6].

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*See equation (2.22) in [5].

†Both $f(r)$ and $g(r)$ contain s as a parameter and can be explicity written as $f(r, s)$ and $g(r, s)$.

‡The deviation integral (12) is the final result obtained by Levine [5] and represents a corrective neutron absorption area of the control rod.

CONCLUSIONS

The analysis presented extends an available steady diffusion solution [5] to the solution of a time-dependent heat conduction problem. In so doing it provides a theoretical basis for the invariability of the rate of loss of heat by conduction in wedges, namely, the difference between the rate of loss of heat per unit time, per unit depth along the two bounding planes, from the wedge surface and that from a corresponding surface of a semi-infinite solid being independent of the time variable. The analysis has found important applications in the area of casting solidification [7, 8].

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HEAT TRANSFER TO HORIZONTAL TUBES IN A PILOT-SCALE FLUIDIZED BED

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NOMENCLATURE

C_R,	correction factor used by Wender and Cooper [8];
D_T,	outer diameter of tube;
$\bar{d}_p$,	surface-to-volume mean particle diameter;
H,	expanded bed height;
H_{mf},	bed height at minimum fluidization;
h_b,	bed-to-exterior time mean heat transfer coefficient;
h_{max},	maximum value of h_b as U increases;
U,	superficial gas velocity;
U_{mf},	superficial gas velocity at minimum fluidization;
ε,	overall bed voidage;
ε_{mf},	bed voidage at minimum fluidization.

INTRODUCTION

WHILE many data have been published on heat transfer between immersed surfaces and gas fluidized beds, there are relatively few results for large-scale systems and for superficial gas velocities typical of industrial beds. During our investigation of heat transfer to horizontal tubes in the freeboard region above fluidized beds [1], some results were also obtained for immersed tubes. These data, reported in this communication, are for a reasonably large column and at gas velocities and temperatures and entrained solids recycle conditions more representative of industrial practice than most previously reported data.

METHODS

The experimental column, constructed of stainless steel, was 0.254 × 0.432 m in cross-section and 3.0 m high. Quartz windows allowed observation and filming of the bed and surface behaviour. The column was heated externally by inconel tubular heaters of total power 21 kW braised to the walls of the column. The operating bed temperature was always in the range 385–425 K. A bundle of horizontal tubes of 25.4 mm o.d. was present in the column in four rows of four tubes each. All tubes were internally finned and made of copper with external chrome-plating to prolong life and reduce absorptivity to radiation. The centre of the lowest row of tubes was 0.76 m above the gas distributor. The vertical